Variance Components Estimate in 2D Geodetic Network

Tomáš BACIGÁL, Michaela KORBAŠOVÁ, Andrej VILLIM and Mária ŽDÍMALOVÁ, Slovakia

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SUMMARY

Current situation in measuring technology, providing high precision / short time measurements, brings up a question of building such a complex mathematic - statistical models that aim at maximally accurate estimate of unknown parameters.

In our paper we focus on estimation of local 2D gedetic network point coordinates and its variances, as well as locally minimum variance quadratic unbiased invariant estimator (LMVQUIE) of range finder. The instrument standard deviation is supposed to be determined by an experiment designed at the waterwork Gabčíkovo. Results are confronted with the net parameters estimate that takes into account those variance components given only by a manufacturer.

Data provided by manufacturer are related to a laboratory temperature (20°C) that is rarely the case outside. The significance of contribution lies in fact, that we estimated the variance components directly from outdoor observations within the mentioned network, thus the instrument's parameters are known for real observational conditions.

This contribution presents the latest theoretical and practical findings in precise local 2D geodetic net constructing and its parameters estimating.

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1. INTRODUCTION

There are many measuring instruments, precision of which can't be described solely by standard deviation σ (Abelovič et al. 1990, Kubáček and Kubáčková 2000). For instance, some electro-optical range finders has it's standard deviation given

$$\sigma_s = a + bs\,, \qquad (1)$$

where s is an observed distance, a and b are precision characteristics. Parameter a is also called additive and b multiplicative constant and can be determined in two following ways:

- we have the opportunity to carry out auxiliary experiment, point of which is to estimate the parameters (on observation base)
- we've got a data file from practical experiment and the parameters come out as by-products from estimation

In our paper we deal with the later alternative using some experimental data obtained at Waterwork Gabčíkovo (WwG).

2. EXPERIMENT DESCRIPTION

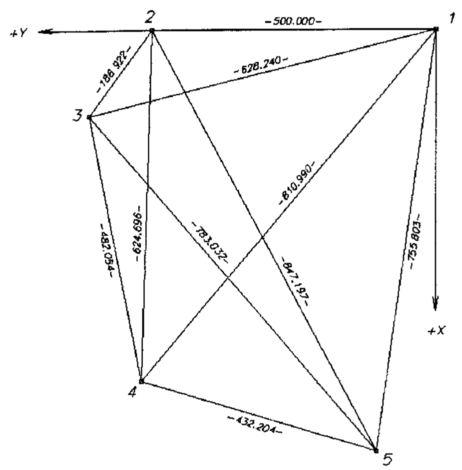

Figure 1: Local geodetic network used for experiment

Measurements was performed within the local geodetic net by WwG, which serves for deformation monitoring and contains 236 deeply stabilised pillars equipped for dependent centering of instruments (Kožár and Lukáč 2000). We picked out 5 verticaly diverse points distanced at most 847m (see Figure 1). Observations lasted no longer than 30 minutes per station, horizontal distances were observed six times with Zeiss Elta Trimble 3602 DR rangefinder. Precision quoted by manufacturer is 2 + 2ppm, observed values are summarized in Table 1.

Table 1: Observed distances

between points		observation number						mean
		1	2	3	4	5	6	
		[m]						
1	2	500.0010	500.0021	500.0012	500.0023	500.0015	500.0020	**500.0017**
1	3	628.2402	628.2413	628.2400	628.2412	628.2413	628.2409	**628.2408**
1	4	810.9908	810.9919	810.9905	810.9916	810.9917	810.9914	**810.9913**
1	5	755.8057	755.8048	755.8059	755.8056	755.8051	755.8043	**755.8052**
2	3	186.9229	186.9218	186.9226	186.9228	186.9215	186.9222	**186.9223**
2	4	624.6967	624.6968	624.6965	624.6954	624.6962	624.6958	**624.6962**
2	5	847.1985	847.1996	847.1993	847.1999	847.1988	847.1993	**847.1992**
3	4	482.0555	482.0567	482.0565	482.0553	482.0560	482.0565	**482.0561**
3	5	783.0330	783.0345	783.0337	783.0343	783.0335	783.0337	**783.0338**
4	5	432.2040	432.2050	432.2044	432.2042	432.2043	432.2054	**432.2046**

3. MATHEMATICAL MODEL

The purpose of our model for 2D geodetic network is to estimate all the unknown coordinates of network points and to determine instrument's variance components by the Locally Minimum Variance Quadratic Unbiased Invariant Estimator (LMVQUIE) method.

Deterministic model, briefly denoted

$$\mathbf{X} = \mathbf{f}(\mathbf{\Theta}), \tag{2}$$

where function **f** for measured distance is not linear

$$\mathbf{s}_{kl} = \sqrt{(Y_l - Y_k)^2 + (X_l - X_k)^2}, \tag{3}$$

is to be linearized

$$\mathbf{X} = \mathbf{f}(\mathbf{\Theta}_0) + \mathbf{A}\,\Delta\mathbf{\Theta} \tag{4}$$

and rearranged

$$\mathbf{Y} = \mathbf{A}\,\Delta\mathbf{\Theta}. \tag{5}$$

s is a vector of distances included in vector of observations **X**, Y and X are coordinates, $\Theta = \Theta_0 + \Delta\Theta$ is a vector of unknown parameters, **A** design matrix, $\mathbf{Y} = \mathbf{f}(\Theta_0) - \mathbf{X}$ denotes a known constant vector and k, l indices of the five involved points.

As for dispersion of realized distance $\mathrm{E}(\varepsilon_s^2) = \mathrm{Var}(\varepsilon_s) = \sigma_s^2$, where ε_s is random error, it may be rewriten with respect to (1)

$$\sigma_s^2 = (a_0 + b_0 s)^2 + 2(a_0 + b_0 s)\delta_a + 2s(a_0 + b_0 s)\delta_b, \tag{6}$$

where a_0 = 2mm and b_0 = 2ppm. Let's denote $\upsilon_1 = \delta_a$, $\upsilon_2 = \delta_b$ and because of trilateration network the $\mathrm{Var}(\varepsilon_s)$ is configured

$$\mathbf{\Sigma}_{(\upsilon)} = \mathbf{V_0} + \upsilon_1 \mathbf{V_1} + \upsilon_2 \mathbf{V_2}, \tag{7}$$

where υ_1, υ_2 are precision characteristics of instrument (variance components), $\mathbf{V_0}$, $\mathbf{V_1}$, $\mathbf{V_2}$ diagonal matrices 10×10

$$\begin{matrix} \{\mathbf{V_0}\}_{i,i} = (a_0 + b_0 s)^2 & \{\mathbf{V_1}\}_{i,i} = 2(a_0 + b_0 s) & \{\mathbf{V_2}\}_{i,i} = 2s(a_0 + b_0 s) \\ \{\mathbf{V_0}\}_{i,j} = 0 & \{\mathbf{V_1}\}_{i,j} = 0 & \{\mathbf{V_2}\}_{i,j} = 0 \end{matrix}, \tag{8}$$

i, j = 1,…10 are indices of observables.

Stochastic model has the form

$$\mathrm{E}(\mathbf{X}) = \mathbf{f}(\mathbf{\Theta_0}) + \mathbf{A}\,\Delta\mathbf{\Theta} + \boldsymbol{\varepsilon}, \tag{10}$$

and variances

$$\mathrm{Var}(\mathbf{X}) = \sum_{i=1}^{p} \upsilon_1 \mathbf{V_i} + \mathbf{V_0}, \tag{11}$$

where p is the number of variance components, in our case $p = 2$. Because of first three coordinates having locked ($Y_1 = X_1 = X_2 = 0$ m, see Figure 1) dim(Θ) = 5×2 –3 = 7.

Variance components estimation follows (Rao and Kleffe 1988)

$$\hat{\boldsymbol{\upsilon}} = \mathbf{S}^{-1}_{(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+}(\mathbf{a}-\mathbf{b}), \tag{12}$$

a, **b** are p-dimensional vectors

$$\mathbf{a} = \begin{pmatrix} \mathbf{Y}^{\mathrm{T}}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V_1}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{Y} \\ \vdots \\ \mathbf{Y}^{\mathrm{T}}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V_p}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{Y} \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} \mathrm{Tr}\left[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V_1}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V_0}\right] \\ \vdots \\ \mathrm{Tr}\left[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V_p}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V_0}\right] \end{pmatrix}, \tag{13}$$

$\mathbf{S}^{-1}_{(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+}$ is p×p matrix

$$\left\{\mathbf{S}^{-1}_{(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+}\right\}_{i,j=1\ldots p} = \mathrm{Tr}\left[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\upsilon}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\upsilon}_j\right], \tag{14}$$

$(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+$ represents Moore-Penrose inversion

$$(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+ = \boldsymbol{\Sigma}_\mathbf{0}^{-1} - \boldsymbol{\Sigma}_\mathbf{0}^{-1}\mathbf{A}(\mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_\mathbf{0}^{-1}\mathbf{A})\mathbf{A}^{\mathrm{T}}\boldsymbol{\Sigma}_\mathbf{0}^{-1}, \tag{15}$$

where $\boldsymbol{\Sigma}_\mathbf{0} = \sum_{i=1}^{p}\boldsymbol{\upsilon}_{0i}\mathbf{V_i} + \mathbf{V_0}$,

$\boldsymbol{\upsilon}_0$ are approximate values taken from instrument’s certificate (2mm and 2ppm).
The model is being iterated

$$\hat{\boldsymbol{\upsilon}} = \boldsymbol{\upsilon}_0 + \mathbf{S}^{-1}_{(\mathbf{M}_X\boldsymbol{\Sigma}_{0(\boldsymbol{\upsilon}_0)}\mathbf{M}_X)^+}(\mathbf{a}-\mathbf{b})_{(\boldsymbol{\upsilon}_0)},$$

$$\hat{\hat{\boldsymbol{\upsilon}}} = \hat{\boldsymbol{\upsilon}} + \mathbf{S}^{-1}_{(\mathbf{M}_X\boldsymbol{\Sigma}_{0(\hat{\boldsymbol{\upsilon}})}\mathbf{M}_X)^+}(\mathbf{a}-\mathbf{b})_{(\hat{\boldsymbol{\upsilon}})}, \tag{16}$$

until the $(\hat{\hat{\boldsymbol{\upsilon}}} - \hat{\boldsymbol{\upsilon}}) = 0$ stop criterion.
Then unknown parameter estimation can be expressed

$$\Delta\hat{\mathbf{\Theta}} = (\mathbf{A}^{\mathrm{T}}\Sigma^{-1}_{0(\hat{\hat{\boldsymbol{\upsilon}}})}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\Sigma^{-1}_{0(\hat{\hat{\boldsymbol{\upsilon}}})}\mathbf{Y}, \tag{17}$$

as well as covariance matrix of unknown parameters estimate

$$\mathrm{Var}(\hat{\mathbf{\Theta}}) = \sigma_0^2(\mathbf{A}^{\mathrm{T}}\Sigma^{-1}_{0(\hat{\hat{\boldsymbol{\upsilon}}})}\mathbf{A})^{-1}, \tag{18}$$

where σ_0 is a mean unit error, and covariance matrix of variance components estimates related to the point $\boldsymbol{\upsilon}_0$

$$\mathrm{var}(\hat{\boldsymbol{\upsilon}} \mid \boldsymbol{\upsilon}_0) = 2\mathbf{S}^{-1}_{(\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+} . \quad (19)$$

4. RESULTS

Here we provide estimation outcome involved in LMVQUIE procedure and comparison to a model of bondless network, which doesn't take the variance components estimation into account.

Firstly, the results of iteration (16) are values of variance components and their standard deviations

$\boldsymbol{a = 1.75}$ **mm** $\boldsymbol{\sigma_a = 0.06}$ **mm**

$\boldsymbol{b = 1.92}$ **ppm** $\boldsymbol{\sigma_b = 0.10}$ **ppm**

comparing with factory values for the laboratory temperature +20°C

$a = 2.00$ mm $\sigma_a = 0.20$ mm

$b = 2.00$ ppm $\sigma_b = 0.10$ ppm .

Table 2 shows estimate of coordinates and their precision characteristics by the LMVQUIE method, while Table 5 deals with distances instead. Comparizon to the alternative method with bondless network and alternative coordinates estimates are brought in Table 4 and Table 3, respectively.

Table 2: Coordinates estimates by LMVQUIE

point number	approximate coordinates		estimated coordinates		standard deviations	
	X_0	Y_0	X	Y	σ_X	σ_Y
	[m]				[mm]	
1	0	0	-	-	0	0
2	0	500.0000	-	500.0019	0	0.62
3	151.3135	609.7452	151.3136	609.7463	0.51	0.46
4	624.4515	517.4600	624.4525	517.4610	0.41	0.54
5	748.6833	103.4952	748.6855	103.4965	0.24	0.62

Table 3: Coordinates estimates by alternative method (bondless network)

point number	approximate coordinates		estimated coordinates		standard deviations	
	X_0	Y_0	X	Y	σ_X	σ_Y
	[m]				[mm]	
1	0	0	-	-	0	0
2	0	500.0000	-	500.0014	0	0.62
3	151.3135	609.7452	151.3135	609.7467	0.67	0.69
4	624.4515	517.4600	624.4528	517.4610	0.60	1.14
5	748.6833	103.4952	748.6854	103.4961	0.66	1.28

As seen, the maximum difference is 0.4 mm in coordiantes, and -0.66mm in mean errors. Variances of estimated parameters clearly speak for including the variance components into estimation procedure. The same unequality in favour of LMVQUIE holds if considering Table 5.

Table 4: Standard deviations comparison

point number	LMVQUIE		bondless network		differences	
	σ_X	σ_Y	σ_X	σ_Y	$\Delta\sigma_X$	$\Delta\sigma_Y$
	[mm]		[mm]		[mm]	
1	0	0	0	0	0	0
2	0	0.62	0	0.62	0	0
3	0.51	0.46	0.67	0.69	-0.16	-0.23
4	0.41	0.54	0.60	1.14	-0.19	-0.60
5	0.24	0.62	0.66	1.28	-0.42	-0.66

Table 5: Distances estimates.

obs. number	distance	observed	estimate		standard deviation of estimates	
			LMVQUIE	bondless net	LMVQUIE	bondless net
		[m]			[mm]	
1	1 – 2	500.0017	500.0019	500.0014	0.62	0.62
2	1 – 3	628.2408	628.2407	628.2411	0.49	0.62
3	1 – 4	810.9913	810.9913	810.9915	0.18	0.68
4	1 – 5	755.8052	755.8052	755.8050	0.28	0.75
5	2 – 3	186.9223	186.9215	186.9219	0.72	0.48
6	2 – 4	624.6962	624.6966	624.6968	0.41	0.58
7	2 – 5	847.1992	847.1992	847.1991	0.11	0.66
8	3 – 4	482.0561	482.0550	482.0554	0.51	0.56
9	3 – 5	783.0338	783.0339	783.0343	0.23	0.65
10	4 – 5	432.2046	432.2042	432.2044	0.61	0.62

5. CONCLUSION

All the results achieved by method LMVQUIE indicate we chose an appropriate mathematical model, which has a plus in estimating the variance components from direct observations (no auxiliary procedures are needed), and furthermore valid for particular outer conditions (factory parameters are limited in this way). As the matrix **S** from (14) is conditioned by a spectrum wideness of observed distances, we implemented this precondition into the experiment in order to utilize this modelling method effectively.

The estimated variance parameters of our rangefinder differs from those quoted in its certificate in values 0.25 mm for *a* and 0.08 ppm for *b*, however this is still in '$t\sigma$ criterion' boundaries ($t_{\alpha = 0.05} = 2$).

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BIOGRAPHICAL NOTES

Tomáš Bacigál, Ing., is a PhD student (2nd year) at the Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava.

Michaela Korbašová, Ing., is an assistant professor (4 years) at the Department of Surveying, Faculty of Civil Engineering, Slovak University of Technology in Bratislava.

Andrej Villim, Ing., had been an assistant professor for 4 years at the Department of Surveying, Faculty of Civil Engineering, Slovak University of Technology in Bratislava. At present he works as supervisor at Technical Department at Land Cadastre Administration (Register of property).

Mária Ždímalová, Mgr., is a PhD student (2nd year) at the Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava.

CONTACTS

Ing. Tomáš Bacigál
Faculty of Civil Engineering at Slovak University of Technology
Radlinskeho 11, 813 68
Bratislava
SLOVAKIA
Tel. + 421 2 52961 477
Email: bacigal@math.sk
Web site: www.math.sk/bacigal

Ing. Michaela Korbašová
Faculty of Civil Engineering at Slovak University of Technology
Radlinskeho 11, 813 68
Bratislava
SLOVAKIA
Fax + 421 2 52494 334
Email: michaela.korbasova@stuba.sk

Mgr. Mária Ždímalová
Faculty of Civil Engineering at Slovak University of Technology
Radlinskeho 11, 813 68
Bratislava
SLOVAKIA
Tel. + 421 2 52961 477
Email: zdimalova@math.sk