

# Coordinate Transformations

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## SUMMARY

Local geodetic datums have been developed in the past, in order to satisfy the surveying and mapping requirements of countries all over the earth. The broad use of GPS observations, as part of the surveying routine, has shifted the interest from the local to the world geodetic systems. The transformation of coordinates between geodetic systems has always been of interest, but the new needs have made it more important. Three dimensional similarity transformations are being used in Geodesy in order to transform coordinates between three dimensional geodetic datums, although the two dimensional approach is often followed, especially for small networks.

This paper deals with the congruency of the two approaches as well as with the tolerances with respect to the size of networks, solved within the Hellenic Geodetic Reference System (HGRS '87).

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## 1. INTRODUCTION

A fundamental activity in land surveying is the integration of multiple sets of geodetic data, gathered in various ways, into a single consistent data set, that is into a common geodetic reference frame. In the past it was sufficient, but in some cases also unavoidable, to combine all such data using a locally, mostly arbitrarily, defined geodetic datum.

In recent years, a growing trend toward the use of satellite positioning and global mapping satellite systems has been developed providing position – based products in a world reference frame. One of the principal purposes of such a world frame is to eliminate the use of multiple geodetic datums. But, until such a world geodetic reference frame is accepted, used and implemented worldwide, the satellite data may lead to several practical difficulties when the results need, also, to be related to a geodetic datum, as is often the case. Such problems arise in several instances, such as navigation, revision of older maps, cadastral surveying, industrial surveying, deformation studies, geo-exploration etc.

In general, the necessity of transforming data from one reference frame to another is solved by applying a coordinate transformation. Although coordinate transformations are straightforward mathematically, they may cause several problems when applied, for various reasons, such as poor knowledge of the distortions and inconsistencies of the local datum, or even lack of sufficient knowledge of geodesy of people who use such transformations.

To properly understand coordinate transformations in geodesy, it is essential to understand the relationship between a geodetic reference system, which is mathematically established, and its realisation, via geodetic observations, the GRF (*Geodetic Reference Frame*). Naturally, the GRF has some degree of uncertainty, due to observational errors in the determination of the coordinates of the ground points.

Often, there are two GRFs realising two different systems (local or global), with a number of common network points. Since the GRFs are not perfect realisations of their systems, only a *best estimate* of the transformation parameters with their respective standard deviations may be computed. In practice, this means that no exact transformation exists between two geodetic coordinate systems. The degree of inconsistency in a transformation will depend on the patterns of errors present in the two GRFs, which often are a result of the geodetic methods used to establish the GRF, and also on how carefully the transformation has been designed to take into account those errors.

- The choice of the appropriate coordinate transformation model depends on factors such as:
- The size of the area (sub - network) of interest and the distortions of the local datum in this area
- The type of the network (3D or 2D or even 1D) and the accuracy required

The present work deals with the estimation of the parameters of the similarity transformation model and their application to geodetic networks. Both the three dimensional and the two dimensional transformations are discussed and the congruency of the two approaches is investigated with respect the size of the network. Special care is given to the case when older geodetic coordinates, referring to a geodetic datum, are compared with recent ones, referring to a world reference frame, for monitoring deformation.

## 2. THE MODELS

The transformation of coordinates from one geodetic frame to another is possible with the coordinates of points expressed in  $(\varphi, \lambda, h)$ . The formulae are known as the *Full Molodensky formulae* or the *Abridged Molodensky* ones, the latter having a lower degree of accuracy. These formulae deal only with a translation of the origin and changes in ellipsoid size and shape, while a difference in orientation of the ellipsoidal axes is not included.

The most general transformation model is the *affine transformation*, where changes in position, size and shape of a network are allowed. The scale factor of such a transformation depends on the orientation but not on the position within the net. Hence the lengths of all lines in a certain direction are multiplied by the same scalar.

A transformation in which the scale factor is the same in all directions is called a *conformal* or *similarity transformation*. It preserves shape but not size. An *orthogonal transformation* is a similarity transformation in which the scale factor is unity. In this case the shape and size of the network will not change, but the positions of points do.

### 2.1 3D Transformation Models

Six parameters are needed to describe the relation between two geodetic reference frames, three translation parameters and three rotations between the coordinate axes. Thus, strictly speaking, no scale distortion should be considered as part of a coordinate transformation, since a scale difference represents a systematic distortion of positions (coordinates) rather than of the reference frame itself. [Vanicek P., Steeves R.R., 1996]

Transformation parameters may have universal, national or local character. Thus, universal or global model parameters, relating two homogeneous satellite datums, are derived from globally distributed network data. National geodetic datum parameters are usually derived by national geodetic authorities and typically relate the national geodetic datum to a world frame, for example HGRS '87 (Hellenic Geodetic Reference System '87) and WGS84. There are, also, cases where parameters are determined locally for a limited area (sub-network), such as when a recent GPS survey is to be incorporated to a national network, or a GPS survey is compared with older geodetic data for monitoring deformations.

Since, in most cases, only a limited number of common network points are available, the similarity transformation is preferred over the other transformation models due to the simplicity of the model.

The similarity transformation is efficient for relating 3-D GPS networks to terrestrial networks, although using such a transformation on a large network may distort local scale

and orientation, since it will tend to smooth out local distortions. In order to overcome this, it may be more appropriate to divide a region into smaller zones, each with their own set of transformation parameters. Presumably, the results of the transformation may be unreliable outside the area spanned by the common points used to derive the transformation parameters.

When coordinate transformations between geodetic reference frames are applied, small values are expected for the rotation and the scale parameters. Thus, assuming rotation parameters of the order of a few seconds of arc, the following form of the 3D similarity transformation is often used, known as the *Bursa-Wolf model* [King R.W. et al., 1985]:

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = (1+k) \cdot \begin{bmatrix} 1 & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \quad \text{or} \quad (2.1)$$

$$\vec{X}_2 = (1+k) \cdot R \cdot \vec{X}_1 + \vec{t}_x$$

The transformation includes three translation components,  $t_x$ ,  $t_y$ ,  $t_z$ , three small rotations  $\varepsilon_x$ ,  $\varepsilon_y$ ,  $\varepsilon_z$  and a scale component  $k$ , which is the deviation of the scale from unity ( $1+k$ ) and is small enough to be expressed in ppm.

This model is used when global or national geodetic datum transformation parameters are to be estimated. In the case of networks over limited areas the rotation parameters are highly correlated with the translation ones, due to the large distances between the implied origin of the geodetic datum and the positions of the network points on the earth's surface. Thus, the very small rotations around the coordinate axes are almost indistinguishable from additional translations, resulting in translation components significantly different from the "national" ones, representative for the whole geodetic datum.

In order to overcome this problem an alternative of the above formula is used, relating the transformation parameters to the center of mass of the network – or any convenient point ( $X_0$ ,  $Y_0$ ,  $Z_0$ ) within the network - and the model becomes:

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} k & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & k & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & k \end{bmatrix} \cdot \begin{bmatrix} X_1 - X_0 \\ Y_1 - Y_0 \\ Z_1 - Z_0 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \quad (2.2)$$

Since the rotational and scale parameters depend on the relative positions (baseline vectors) the small rotations and the scale component are the same for both (2.1) and (2.2), but the translation parameters are not.

The minimum number of common points, in order to solve for the seven transformation parameters, is three, although, more points are often available and a least squares solution is applied. The significance of the estimated transformation parameters should be evaluated.

## 2.2 2D Transformation Model

In cases of relatively small networks, (less than 100kmH100km), another approach is often used. The initial 3D Cartesian coordinates are converted to geodetic ones and, finally, to map projection coordinates. Then, the full 2D similarity transformation, with two translation parameters  $\Delta x_0$ ,  $\Delta y_0$ , one rotation  $\theta$  and a scale parameter  $\mathbf{K} = (1+k)$ , known as **Helmert transformation**, may be expressed as [Hofmann-Wellenhof B. et als, 2001]:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Delta x_0 \\ \Delta y_0 \end{bmatrix} + \mathbf{K} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad (2.3)$$

According to a slightly different approach the 3D translation parameters ( $\Delta X_0$ ,  $\Delta Y_0$ ,  $\Delta Z_0$ ) between the two sets of coordinates are computed and applied. Both sets of coordinates are projected to a common reference ellipsoid and then converted to projection coordinates. Finally, the parameters of a full 2D similarity transformation, including translation components, are estimated. Since the projection of the center of mass of the set of network points in the 3D space does not coincide with the center of mass of the projections of the same points, additional translation components in the 2D space should be estimated.

The following linear expression for the Helmert transformation is often used:

$$\begin{aligned} x_2 &= ax_1 - by_1 + \Delta x_0 \\ y_2 &= bx_1 + ay_1 + \Delta y_0 \end{aligned} \quad (2.4)$$

where  $\mathbf{a} = \mathbf{K}\cos\theta$  and  $\mathbf{b} = \mathbf{K}\sin\theta$ .

Then the scale parameter and the rotation are, respectively:  $\mathbf{K} = (\mathbf{a}^2 + \mathbf{b}^2)^{1/2}$  and  $\theta = \mathbf{atan}(\mathbf{b}/\mathbf{a})$ .

## 3. PROCEDURE

In the present work, two average size networks (about 100kmH100km) were considered. The first one was part of a GPS network established in the Corinthian gulf for monitoring the tectonic behaviour [Briole P., et als, 2000], while the second one was a sub-network of a similar GPS network in the area of Euboea [Veis G., et als, 1999]. Also, a case was considered where the points of both networks were taken into account (250kmH150km). Finally, the case of a small network (10kmH10km) in the vicinity of the Corinthian gulf was examined. For all network points two sets of coordinates were available; one HGRS '87 and one set expressed in ITRF 2000. The transformation parameters for transforming from HGRS '87 to ITRF 2000 were sought after in all cases.

The GPS observations for the Corinthian gulf network were carried out in epoch 1995.8 and the ones for the Euboea network in 1997.7. On the other hand, the ground observations used for the computation of the HGRS '87 coordinates took place around 1970. Thus, for both

areas with active tectonic behaviour, there was an interval of about 30 years between the two sets of coordinates.

It should be mentioned that, when GPS data are compared with older geodetic data for monitoring deformations, the two sets of coordinates show significant discrepancies and it is rather difficult to distinguish which part is due to the non coincidence of the reference frames and which is due to real displacements.

In order to avoid this problem and to test the software developed, a simulation was carried out, initially, considering two sets of coordinates. The first one was the ITRF 2000 set from the Corinthian gulf, while the second one was formed from the first one after applying a specific 3D similarity transformation and random noise. Thus, the second set was very close to an HGRS '87 set, but without the inconsistencies of a real data set, as mentioned above (*Table 1, col. 1*).

In the case of the 3D similarity transformation both formulae, (2.1) and (2.2), were considered. The sub matrices  $A_i$  of the design matrix  $A$  of the least squares observation equations for the estimation of the transformation parameters vector

$\hat{x} = (k \ \varepsilon_x \ \varepsilon_y \ \varepsilon_z \ t_x \ t_y \ t_z)^T$  are of the form:

$$A_i = \begin{pmatrix} X_i & 0 & -Z_i & Y_i & 1 & 0 & 0 \\ Y_i & Z_i & 0 & -X_i & 0 & 1 & 0 \\ Z_i & -Y_i & X_i & 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.5)$$

for formula (2.1) and of the form:

$$A_i = \begin{pmatrix} (X - X_0)_i & 0 & (-Z + Z_0)_i & (Y - Y_0)_i & 1 & 0 & 0 \\ (Y - Y_0)_i & (Z - Z_0)_i & 0 & (-X + X_0)_i & 0 & 1 & 0 \\ (Z - Z_0)_i & (-Y + Y_0)_i & (X - X_0)_i & 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.6)$$

for the case of (2.2), while the right hand vector for both cases is of the form:

$$l_i = \left( (X_2 - X_1)_i \ (Y_2 - Y_1)_i \ (Z_2 - Z_1)_i \right)^T$$

It should be mentioned that, for both cases, there is a marked difference in magnitude between the coefficients in the sub matrices  $A_i$  (2.5 and 2.6), which also affects the compatibility of the significant digits in the elements of the normal equations matrix.

In order to overcome this problem the typical procedure was slightly altered by estimating first the 3D translation parameters, applying these values to the initial HGRS '87 coordinate set and, finally, estimating only the three small rotations  $\varepsilon_x$ ,  $\varepsilon_y$ ,  $\varepsilon_z$  and the scale component  $k$  in (2.1). According to this approach, the least squares solution becomes more stable, since the

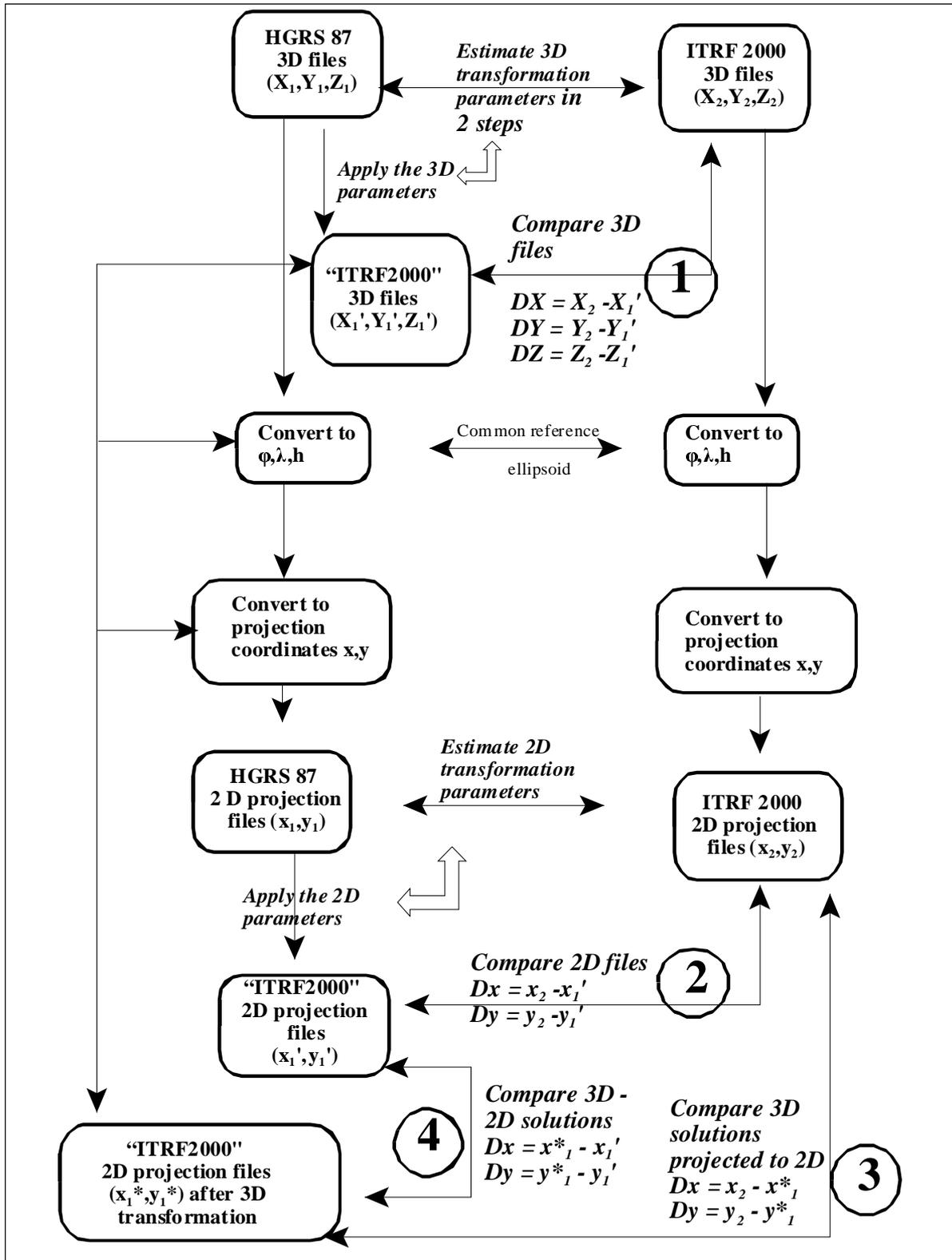
coefficients of the normal equations matrix are of the same order of magnitude. Also, in this case, although the translation parameters do not vary from the ones in the rigorous solution, the scale and the rotation components become very small, without the need for further iterations.

The steps followed are depicted in *Figure 1* (case **1**), while the results of this approach for all cases examined are presented in *Table 1*.

In this case of the 2D approach the linear model described in (2.4) was considered, after the conversion of the initial data sets to the common reference ellipsoid of GRS 80, and then to the projection coordinates. The approach of estimating first the translation parameters in the 3D space and then solving for the full 2D transformation model was also considered, in some cases. The steps of the standard procedure are depicted in *Figure 1* (case **2**), while the results of both procedures are, also, presented in *Table 1*.

In all cases, and for both the 3D and 2D approach, the computed best estimates of the transformation parameters were applied to the initial HGRS '87 coordinate sets and the results were compared with the ITRF 2000 sets, in order to derive the discrepancies between the pairs of coordinate sets (*Table 2*).

In the case of the three dimensional transformation not only the ( $\delta X$ ,  $\delta Y$ ,  $\delta Z$ ) discrepancies were computed (*Table 2*, case **1**), but the resultant transformed HGRS '87 files were converted to the projection (*Figure 1*, case **3**) and compared with the respective two dimensional ITRF 2000 ones (*Table 2*, case **3**).



**Figure 1** Flow chart of the approaches followed for both the 3D and 2D transformation models

	Parameters	Networks				
		Simulated	Corinth	Euboea	Large	Small
<b>3D Solutions</b>	$\Delta X_0 \pm \sigma_X$ (m)	-201.440±.003	-201.444±.208	- 199.959±.054	-200.609 ±.161	-200.744±.276
	$\Delta Y_0 \pm \sigma_Y$ (m)	74.270±.003	74.260 ±.208	74.842 ±.054	74.587 ± .161	74.520± .276
	$\Delta Z_0 \pm \sigma_Z$ (m)	245.418±.003	245.413±.208	246.214±.054	245.863±.161	245.544± .276
	$k \pm \sigma_k$ (ppm)	0.0±0.0	0.0±0.03	0.0±0.0	0.0±0.02	0.0±0.02
	$\varepsilon_x \pm \sigma_{\varepsilon_x}$ (")	0.0±0.02	0.94±1.2	0.59±0.13	0.61±0.40	2.18± 7.6
	$\varepsilon_y \pm \sigma_{\varepsilon_y}$ (")	0.0±0.01	0.39±0.49	0.26±0.05	0.26±0.17	0.89 ± 3.11
	$\varepsilon_z \pm \sigma_{\varepsilon_z}$ (")	0.0±0.02	0.80±1.02	0.51±0.11	0.52±0.34	1.84 ± 6.45
<b>2D Solutions</b>	$\Delta x_0 \pm \sigma_X$ (m)	148.729±.940	120.185±15.869	132.694±3.152	131.992±3.456	89.273± 2.432
	$\Delta y_0 \pm \sigma_Y$ (m)	309.340±.940	292.535±15.869	303.605±3.152	309.750±3.456	322.510± 2.432
	$k \pm \sigma_k$ (ppm)	5.0±0.22	0.51±3.7	3.4±0.74	4.8±0.81	7.1±2.9
	$\omega \pm \sigma_\omega$ (")	-0.17±0.05	-1.48±0.77	-0.91±0.15	-0.97±0.17	3.10±.61
<b>2D solutions after 3D translation</b>	$\Delta x_0 \pm \sigma_X$ (m)			-17.853±3.162		-60.127±14.294
	$\Delta y_0 \pm \sigma_Y$ (m)			-6.348±3.162		18.590±14.294
	$k \pm \sigma_k$ (ppm)			1.9±1.05		4.3±3.4
	$\omega \pm \sigma_\omega$ (")			4.0±1.05		-14.5±3.4

**Table 1** Transformation parameters for both 3D and 2D models with respective r.m.s.

In the case of the two dimensional transformation the transformed projection coordinates of HGRS '87 were compared with the also projected ITRF 2000 ones (*Table 1*, case 2). When a three dimensional translation was first estimated and then the full 2D model was evaluated, the comparison was made between the two dimensional coordinate sets (*Table 2*).

Finally, discrepancies were computed between the 2D transformed HGRS '87 sets, after the application of a 2D transformation, and the same HGRS '87 sets after a 3D transformation and consequent conversion to projection coordinates (*Figure 1*, case 4), (*Table 2*, case 4).

#### 4. DISCUSSION - CONCLUSIONS

In the case of the simulated network (*Table 2*, col. 1), all discrepancies computed in three and two dimensions were less than 2-3cm, of the same order as the least squares residuals in the estimation of the transformation parameters. This good agreement is achieved as the discrepancies between the pairs of coordinate sets are only due to random errors.

In all other cases and irrespectively of the method used for the estimation of the transformation parameters, the size of the discrepancies between the pairs of coordinate sets, as well as of the respective residuals, is significant (of the order of several tenths of centimetres), due to the presence of a displacement field both in the Corinthian gulf and the vicinity of Euboea (*Table 2*, col. 2-4).

Types of solutions	Range of discrepancies in cm				
	<i>Simulated Network</i>	<i>Corinth</i>	<i>Euboea</i>	<i>Large Network</i>	<i>Small Network</i>
<b>3D</b> solution in two steps (case 1)	1-2.5	3-115	1-33	3.5-170	1-155
<b>2D</b> solution (case 2)	1-2.5	1-34	1-21	1-65	0-6.5
<b>3D</b> solution projected to <b>2D</b> (case 3)	1-2	5-48	2-27	2-60	1-19
Comparison <b>3D</b> – <b>2D</b> solution (case 4)	1-2	2.5-29	1-28	1-40	1-13
<b>2D</b> solution after <b>3D</b> translation			1-21		1-8

**Table 2** Range of discrepancies in cm, between the transformed HGRS 87 files and the ITRF 2000 ones, for all networks and all types of transformation models.

It should be mentioned that, in the case of the small network (*Table 2*, col. 5), the discrepancies computed from a 3D solution are quite large on the contrary to the ones derived from the two dimensional models. This is in good agreement with the accepted concept that two dimensional transformation models are more appropriate for small networks.

The above mentioned comparisons indicate that, if local transformation parameters between two geodetic frames are to be estimated, it may be irrelevant whether the 3D or the 2D approach is preferred, even for a relatively large network.

In the case of a 3D transformation it is recommended that the procedure in two steps should be preferred, since the least squares solution becomes more stable and, due to the small values of the rotation and scale parameters, no iterations appear to be necessary.

Another also acceptable approach is to solve for the translation parameters in three dimensions, and then for the full two dimensional model. Obviously, the 2D approach for estimating the transformation parameters is the appropriate choice in the case of small networks.

However, when deformation processes take place in an area, the long term behavior is, often, investigated with geodetic methods, by comparing older and recently acquired data, such as

GPS observations. During the analysis, pairs of two dimensional coordinate sets are, usually, compared in the projection plane and the displacements for the time interval between the two epochs are derived. Such coordinate sets refer, probably, to different reference frames; the older ones to a geodetic datum, such as the HGRS '87 and the recent ones to a world one, such as the WGS 84 or an ITRFxx. Therefore, an estimation of local transformation parameters should take place before the deformation analysis.

Since, usually, the two sets of coordinates refer to different epochs, the size of the residuals in the least squares solution for the transformation parameters, whether the 2D or the 3D approach is preferred, depends largely on the time interval elapsed and the size of the displacement field expected. Consequently, the question of choosing the appropriate procedure in the case of deformation analyses is not easily answered.

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