Optimal Position of Totalstations for Length Measurements of a Chord

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Key words: surveying engineering, electronic distance measurement (EDM), eccentric distance measurement, geometric optimisation, optimal totalstation positioning.

SUMMARY

We determine the optimal position for maximal accuracy when measuring the distance between two points A and B, considering all possible positions of the totalstation.

Our results are obtained using a fully exact mathematical analysis, and lead to the surprising result that the optimal position can occur in four essentially different cases: it may coincide with one of the points A or B, or with the middle point between them, or be situated on the middle perpendicular of the chord AB, or even be situated asymmetrically, away from the middle perpendicular and from the line AB.

Statistically, the distance measurement error of totalstations is the superposition of a fixed error term a, due to the limitations of the electronic timing, and a term bd, proportional to the measured distance d, and due to small fluctuations of the quartz crystal frequency used for modulating the laser. Many manufacturers rate the resulting error as $a+bd$, but both contributions are in fact statistically independent, and hence should be combined using the formula $\sigma_d = \sqrt{a^2 + b^2 d^2}$.

A fundamental mathematical analysis of the error using the right formula will allow us to distinguish four different fundamental cases of optimal location and eleven different totalstation locations due to the geometric symmetry of the problem, depending on the magnitudes of the parameters a and b and the accuracy of angular measurements, c. The four cases can even occur for the same instrument, depending on the length d to be measured. Most surprisingly is the fact that these different cases occur for very common values of a, b and c.

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1. INTRODUCTION

The study of measurement errors using purely optical surveying equipment has already been carried out in detail long ago [Kahmen H. and Faig W. 1988, pp. 150--152 and 177—183]. But nowadays a different measurement technology is widely used by so-called `totalstations' that offer electronic measurement of distances. This can be implemented e.g. by time-lag measurements of infra-red pulses, or by a continuous phase difference analysis of a modulated infra-red wave [Späth H. 1996, Day N.B. 1990].

Statistically, the distance measurement error of such devices is the superposition of a fixed error term a, due to the limitations of the electronic timing, and a term bd, proportional to the measured distance d, and due to small fluctuations of the quartz crystal frequency.

Many manufacturers rate the resulting error as $a+bd$, but both contributions are in fact statistically independent, and hence should be combined using the formula $\sigma_d = \sqrt{a^2 + b^2 d^2}$ [Baumann E. 1993, Brinker R. Minnick R. 1995].

When the distance from a point A to a point B is to be measured electronically to high precision, several strategies can be followed:

1. one might locate oneself at A and measure directly to B;
2. or locate at the point C half-way between A and B and measure the distances AC and BC;
3. or locate at some other point on the perpendicular to AB and through C, and similarly measure the distances to A and B and the angle $A\hat{C}B$, then apply the cosine rule to obtain the distance AB.

These various alternatives have been recommended by different authors [Bannister A., Raymond S., Baker R. 1992, Baumann E. 1993, Brinker R. Minnick R. 1995, Kahmen H. and Faig W. 1988, Methley B.D.F. 1986]. The first one is a direct measurement, the others are eccentric measurements. It will turn out from our computations that in some realistic situations none of these alternatives is actually optimal, the best choices of M lying neither on the line AB nor on its middle perpendicular.

Independently of the mentioned error $\sigma_d = \sqrt{a^2 + b^2 d^2}$ on distance measurements (for certain instrument-dependent values A and B), there is also an error on angular measurements θ, which can be assumed independent, i.e. $\sigma_\vartheta^2 = c^2$, c being an instrument constant.

The three precision parameters *a*, *b* and *c* are positive. Realistic values for them are

a = 2 mm (more generally: 1 mm < a < 5 mm).
b = 3 mm/km (more generally: 1 mm/km < b < 5 mm/km).
c = 1” = one arc second (more generally: 0.5" < c < 20”).
d = 75 m (more generally: 1.5 m < d < 1500 m).

Let *M* be the point at which the measurements are performed, and write d_1 for the distance MA and d_2 for MB; furthermore, let θ stand for the angle $A\hat{M}B$ and d for the distance *AB*, cf. also Figure 1.

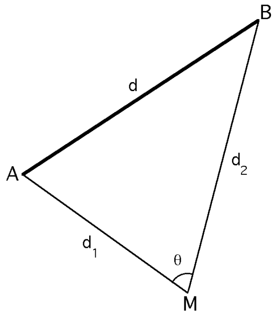

Figure 1. Sketch of the points A, B and totalstation point M

The relationship between these quantities is the cosine rule:

$$d = \sqrt{d_1^2 + d_2^2 - 2d_1 d_2 \cos\theta} \qquad (1)$$

Since the occurring variances are statistically independent and of very small influence on d, one can approximate d linearly near its value and adopt the following variance on it:

$$\sigma_d^2 = \left(\frac{\partial d}{\partial d_1}\right)^2 \sigma_{d_1}^2 + \left(\frac{\partial d}{\partial d_2}\right)^2 \sigma_{d_2}^2 + \left(\frac{\partial d}{\partial \theta}\right)^2 \sigma_\theta^2 \qquad (2)$$

Substituting (1) into (2) leads to the expression:

$$\sigma_d^2 = \frac{(d_1 - d_2\cos\theta)^2 (a^2 + b^2 d_1^2) + (d_2 - d_1\cos\theta)^2 (a^2 + b^2 d_2^2) + d_1^2 d_2^2 \sin^2\theta c^2}{d_1^2 + d_2^2 - 2d_1 d_2 \cos\theta} \qquad (3)$$

in which $\sin^2\theta$ can be rewritten as $1-\cos^2\theta$, and $\cos\theta$ can be expressed in terms of d, d_1 and d_2 by solving (1) for it (when d_1 and d_2 do not vanish):

$$\cos\theta = \frac{d_1^2 + d_2^2 - d^2}{2d_1 d_2} \qquad (4)$$

The resulting expression only involves d_1, d_2, a, b, c and d. Upon expansion of the powers and normalization of the fraction, it appears that d_1 and d_2 only occur in the combinations $d_1^2 + d_2^2$ and $(d_1^2 - d_2^2)^2$. This motivates us to introduce new, dimensionless variables that can replace d_1 and d_2 completely: $T = \frac{2(d_1^2 + d_2^2)}{d^2} - \frac{(d_1^2 - d_2^2)^2}{d^4} - 1$ and $U = \frac{(d_1^2 - d_2^2)^2}{d^4}$.

A general ordered pair *(T,U)* defines uniquely four points, with coordinates $\left(\frac{d}{2}\left(1 \pm \sqrt{U}\right) \pm \frac{d}{2}\sqrt{T}\right)$ in the Cartesian coordinate system for which A has coordinates (0,0) and B (d,0). Clearly, T and U must be nonnegative to define real points. These four points are equivalent for the problem at hand, since (3) can be rewritten in terms of T and U as

$$\sigma_d^2 = \left(1 + \frac{(1-U)^2 - T^2}{(T+U)^2 + 2(T-U) + 1}\right)a^2 + \frac{U+1}{2}b^2d^2 + \frac{T}{4}c^2d^2 \qquad (5)$$

It is also advantageous to introduce the positive dimensionless parameters

$$\begin{cases} \beta = bd/a \\ \gamma = cd/a \end{cases} \qquad (6)$$

so that the variance that should be minimised can be expressed in dimensionless form as

$$\frac{\sigma_d^2}{a^2} = \left(1 + \frac{(1-U)^2 - T^2}{(T+U)^2 + 2(T-U) + 1}\right) + \frac{U+1}{2}\beta^2 + \frac{T}{4}\gamma^2 \qquad (7)$$

Now the problem can be restated as the minimisation of (7) given fixed *ß* and *γ*, over all possible nonnegative T and U. Note that the parameters *ß* and *γ* depend not only on the instrument's precision parameters a, b and c, but also on the distance to be measured, d.

2. MINIMISING THE VARIANCE

The (T,U) domain consists of all nonnegative values, so its boundary, where T=0 or U=0, will have to be inspected separately.

If we consider a ray (T,rT) inside the domain, where r>0 is fixed and T>0 varies, then

$$\frac{\sigma_d^2}{a^2} = \left(1 + \frac{(1-rT)^2 - T^2}{(1+r)^2T^2 + 2(1-r)T + 1}\right) + \frac{rT+1}{2}\beta^2 + \frac{T}{4}\gamma^2 \qquad (8)$$

in which the first term tends to a finite limit when T → ∞, but the second and third one tend to +∞. Consequently, we can restrict ourselves inside the domain to the stationary points.

Now we compare the various measurement locations.

2.1 Case I: direct measurement

Measuring d directly from A to B leads to a dimensionless variance of

$$\frac{\sigma_d^2}{a^2} = 1 + \beta^2 \tag{9}$$

Note that this follows directly from the assumed error behaviour for distance measurements, not from (7), since no angle is measured and no cosine rule is involved. This value also assumes that there is no error in locating oneself at A; if one were to locate near to A but have to measure d_1, d_2 and θ to correct for the difference, the cosine rule would apply and the value would be, from (3) (the formula (7) cannot be used here because (4) does not hold when d_1=0),

$$\frac{\sigma_d^2}{a^2} = 1 + \cos^2\theta + \beta^2$$

so that the best result is to be obtained when θ is 90 degrees. The value then coincides with that in (9).

2.2 Case II: measuring on the line *AB*

The line AB corresponds to the equation T=0. Substituting into (7) gives

$$\frac{\sigma_d^2}{a^2} = 2 + \frac{U+1}{2}\beta^2$$

The minimum is clearly attained for U=0, i.e., the *middle point* of A and B. The value of the dimensionless variance is then

$$\frac{\sigma_d^2}{a^2} = 2 + \frac{1}{2}\beta^2 \tag{10}$$

so that this point is sometimes better than a direct measurement, depending on whether or not ß > √2.

2.3 Case III: measuring on the middle perpendicular to *AB*

The middle perpendicular corresponds to the equation U=0. Substituting into (7),

$$\frac{\sigma_d^2}{a^2} = \frac{2}{1+T} + \frac{1}{2}\beta^2 + \frac{T}{4}\gamma^2$$

and equating the derivative w.r.t. *T* to zero, a stationary point is found at $T=2\sqrt{2}/\gamma-1$. If this value is negative, it does not correspond to a real point, and the minimum is attained on the boundary of the line, i.e., the middle point, which has been treated in the preceding case II. The same point corresponds to the case where this value of *T* is zero. But if this value of *T* is positive, i.e., if $\gamma < 2\sqrt{2}$, there does exist a real stationary point, corresponding to $d_1 = d_2 = \frac{d}{\sqrt{\gamma\sqrt{2}}}$, and whose value of the dimensionless variance is (by substitution):

$$\frac{\sigma_d^2}{a^2} = 2 + \frac{1}{2}\beta^2 - 2\left(1 - \frac{\sqrt{2}}{4}\gamma\right)^2 \qquad (11)$$

so whenever it exists it is better than the measurement at the middle point. As $\gamma \to 0$, this point will come into existence and $\frac{\sigma_d^2}{a^2}$ will tend to $\beta^2/2$. A series expansion of $\frac{\sigma_d^2}{a^2}$ near the stationary value confirms that it is a local minimum:

$$\frac{\sigma_d^2}{a^2} = 2 + \frac{1}{2}\beta^2 - 2\left(1 - \frac{\sqrt{2}}{4}\gamma\right)^2 + \frac{\sqrt{2}}{16}\gamma^3\left(T - \frac{2\sqrt{2}}{\gamma} + 1\right)^2 + \ldots$$

Case III is not always better than case I; the condition for its superiority is (cf. the values obtained in (9) and (11)):

$$\beta^2 > 2 - \frac{1}{2}\left(2\sqrt{2} - \gamma\right)^2$$

2.4 Case IV: other stationary points

The stationary points of the dimensionless variance (7) are determined by equating to zero its partial derivatives w.r.t. *T* and *U*, leading to the equations

$$-\frac{2\left(U(T+U)^2 + \left(T^2 - U^2 - 4TU\right) + (2T-U) + 1\right)}{\left((T+U)^2 + 2(T-U) + 1\right)^2} + \frac{1}{4}\gamma^2 = 0 \qquad (12)$$

and

$$\frac{2T\left((T+U)^2 - 2(T-U) - 3\right)}{\left((T+U)^2 + 2(T-U) + 1\right)^2} + \frac{1}{2}\beta^2 = 0 \qquad (13)$$

A detailed mathematical analysis of these equations leads to the following simple necessary and sufficient conditions:

$$\begin{cases} \frac{\sqrt{2}}{2} \le \gamma \le \sqrt{2} \\ \gamma^2 - 1 \le \beta^4 \\ \beta^2 \le \left(\frac{\sqrt{2}}{2}\right)\gamma\left(\gamma - \frac{\sqrt{2}}{2}\right)\left(2\sqrt{2} - \gamma\right) \end{cases}$$

3. SUMMARY AND EXAMPLES

The different optimal cases can be summarized graphically as in Figure 2. Explicitly, we have that the cases are optimal under the following conditions:

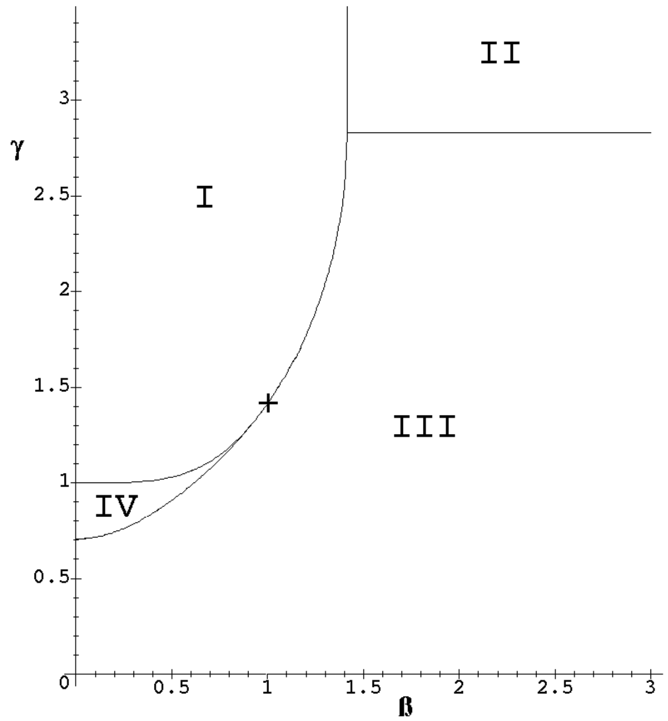

Figure 2: The partition of the (ß,γ) domain.}

3.1 Case I: measurement from an end point

Optimal when

1. $\begin{cases} 2\sqrt{2} \le \gamma \\ \beta \le \sqrt{2} \end{cases}$

2. or when $\begin{cases} \sqrt{2} \le \gamma \le 2\sqrt{2} \\ \beta \le \sqrt{2 - \dfrac{(2\sqrt{2} - \gamma)^2}{2}} \end{cases}$

3. or when $\begin{matrix} 1 \le \gamma \le \sqrt{2} \\ \beta \le \sqrt[4]{\gamma^2 - 1} \end{matrix}$.

3.2 Case II: measurement from the middle point

Optimal when $\begin{cases} 2\sqrt{2} \le \gamma \\ \sqrt{2} \le \beta \end{cases}$

3.3 Case III: measurement from the middle perpendicular

At a point defined by optimal when $d_1 = d_2 = \dfrac{d}{\sqrt{\gamma\sqrt{2}}}$, optimal when

1. $\begin{cases} \sqrt{2} \le \gamma \le 2\sqrt{2} \\ \beta \ge \sqrt{2 - \dfrac{\left(2\sqrt{2} - \gamma\right)^2}{2}} \end{cases}$

2. or when $\begin{cases} \dfrac{\sqrt{2}}{2} \le \gamma \le \sqrt{2} \\ \beta \ge \sqrt{\left(\dfrac{\sqrt{2}}{2}\right)\gamma\left(\gamma - \dfrac{\sqrt{2}}{2}\right)\left(2\sqrt{2} - \gamma\right)} \end{cases}$

3. or when $\gamma \le \dfrac{\sqrt{2}}{2}$.

3.4 Case IV: measurement at another stationary point

Optimal when

$$\begin{cases} \frac{\sqrt{2}}{2} \le \gamma \le \sqrt{2} \\ \gamma^2 - 1 \le \beta^4 \\ \beta^2 \le \left(\frac{\sqrt{2}}{2}\right)\gamma\left(\gamma - \frac{\sqrt{2}}{2}\right)\left(2\sqrt{2} - \gamma\right) \end{cases}$$

In this case, compute $\varepsilon = \sqrt{\frac{\gamma^2}{2-\beta^2}}$, solve

$$-\varepsilon^4 V_1^3 + \varepsilon^2\left(5 + 4\beta^2 - 2\varepsilon^2\right)V_1^2 + \left(16\varepsilon^2 - 4 - 4\beta^4 + 8\varepsilon^2\beta^2 + 8\beta^2\right)V_1 - 8\beta^4 + 16\varepsilon^2 + 16\beta^2 - 8 = 0$$

for V_1, selecting the single root in the interval $0 < V_1 < 2$, then compute

$$V_2 = -V_1 - \frac{V_1^2}{2} + \frac{V_1\sqrt{V_1 + 2}}{\varepsilon}$$

$$U = 1 + \frac{V_1 - V_2}{2}$$

$$T = \frac{V_1 + V_2}{2}$$

$$d_1 = \frac{1}{2}\sqrt{U + T + 1 \pm 2\sqrt{U}}$$

$$d_2 = \frac{1}{2}\sqrt{U + T + 1 \mp 2\sqrt{U}}$$

d_1 and d_2 fix the four equivalent optimal points *M* completely. All square roots are guaranteed to be taken on positive values; the third-degree equation is guaranteed to have precisely one root in the prescribed interval.

This exhausts all possibilities.

4. CLASSIFICATION OF INSTRUMENTS

Recall that ß and γ are proportional to the measured distance (cf. equation (6)) *d*, but that their quotient *γ/ß=c/b* only depends on the instrument. For a given instrument, the possible values of ß and γ therefore lie on a straight line in Fig. 2, through the origin, with slope *c/b*.

In view of the tangent of the three curves joining at the point *(1,√2)* in Fig. 2 having slope √*2*, which coincides with the slope of the line joining this point to the origin, we can classify as follows:

1. If $\frac{c}{b} \leq \sqrt{2}$, case III is optimal for small *d* and case II for large.

2. If $\sqrt{2} \leq \frac{c}{b} \leq 2$, case III is optimal for small *d*, then case IV, then case I, then case III again, then case II.

3. If $2 \leq \frac{c}{b}$, case III is optimal for small *d*, then case IV, then case I, then case II.

5. COMMENTED EXAMPLE

The computations are directly relevant to practice as will be illustrated by the example we will analyse.

Suppose that

a = 2 mm
$b = 3.10^{-6}$ (= 3 mm/km or 3 ppm (parts per million))
c = 1" (= $4{,}848.10^{-6}$ radians)

These are realistic values for a totalstation of geodetic accuracy.

As $c/b = 1{,}616$, $\sqrt{2}<c/b<2$, the totalstation considered belongs to instruments of the second type, where the optimum location varies as a function of the distance measured.

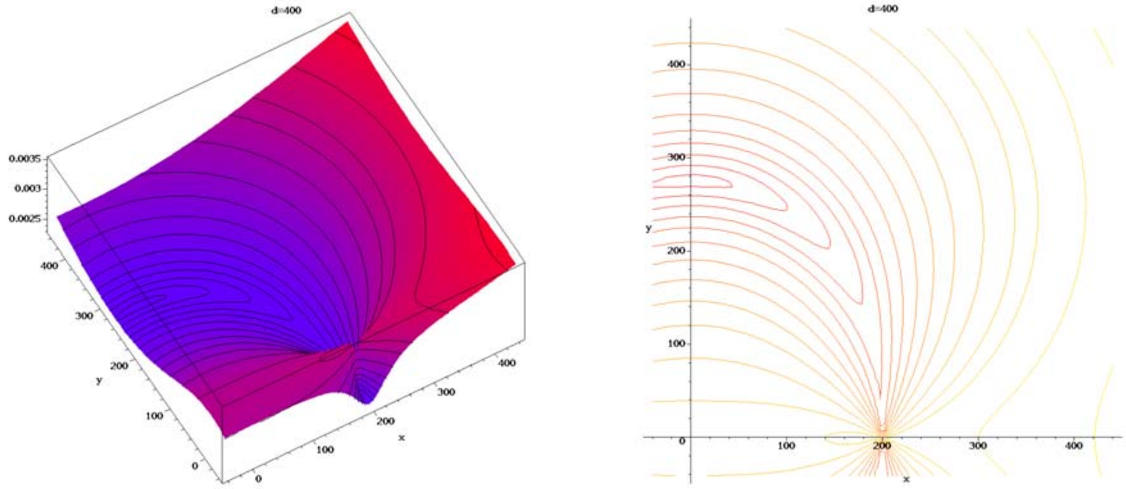

Figure 3

More precisely, different optimal locations are found for distances of 400, 450, 500, 1000 and 1200 m:

a) on the perpendicular of the chord *AB* (case III) (cf. Fig. 3) for *d* = 400 m
b) in a particular point (case IV) (cf. Fig. 4) for *d* = 450 m
c) at the end of the chord (case I) (cf. Fig. 5) for *d* = 500 m
d) on the perpendicular (case III) (cf. Fig. 6) for *d* = 1000 m
e) in the middle of the chord (case II) (cf. Fig. 7) for *d* = 1200 m

As a consequence of the symmetry of the below mentioned geometric problem, there are different optimal locations for each case. Considering an orthogonal two-dimensional reference system with the origin in the middle of the chord, an X-axis that coincides with the chord to measure and an Y-axis that coincides with the perpendicular to the chord, we account for:

a) Case I (endpoint of the chord):	2 solutions (symm. rel. to Y-axis)
b) Case II (middle of the chord):	1 solution
c) Case III (on the perpendicular):	2 solutions (symm. rel. to X-axis)
d) Case IV (particular point):	4 solutions (symm. rel. to. X- & Y-axis)

This yields the following optimal locations for our example:
a) for d = 400 m: case III: (0;-276,83) or (0;276,83) (cf. Fig. 3)
b) for d = 450 m: case IV: (-215,90;-91,66) or (-215,90;91,66) or (215,90;-91,66) or (215,90;91,66) (cf. Fig 4)
c) for d = 500 m: case I: (-250;0) or (250;0) (cf. Fig. 5)
d) for d = 1000 m: case III: (0;-204,23) or (0;204,23) (cf. Fig. 6)
e) for d = 1200 m: case II: (0,0) (cf. Fig. 7)

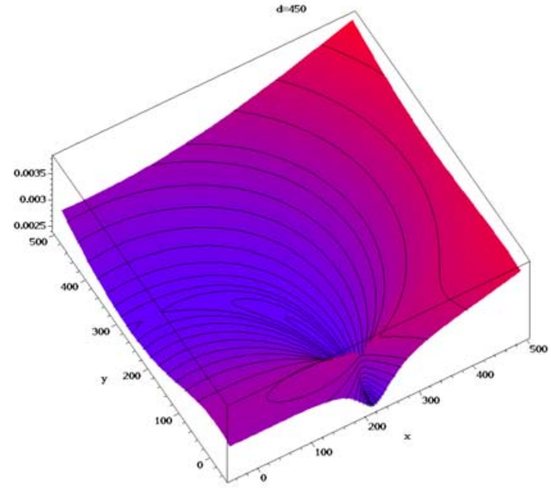

Figure 4

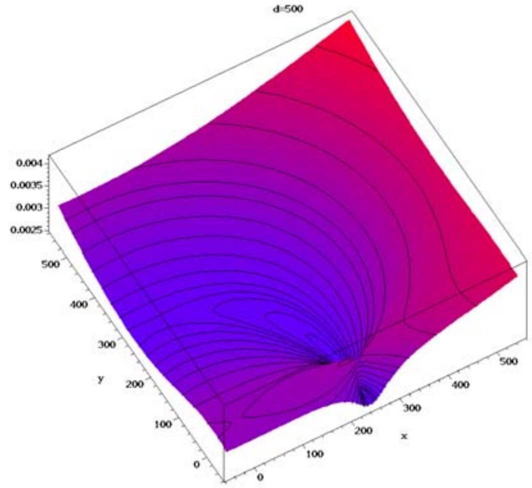

Figure 5

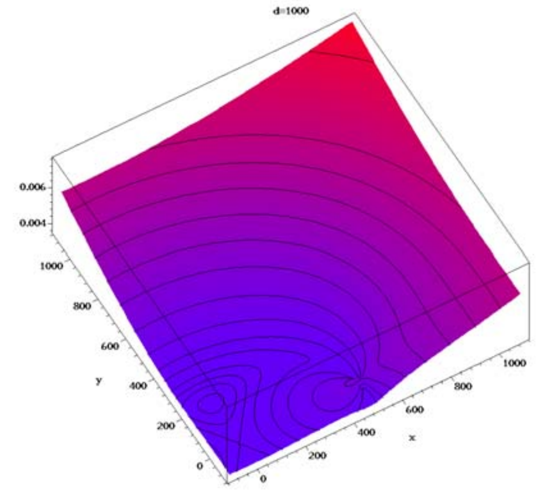

Figure 6

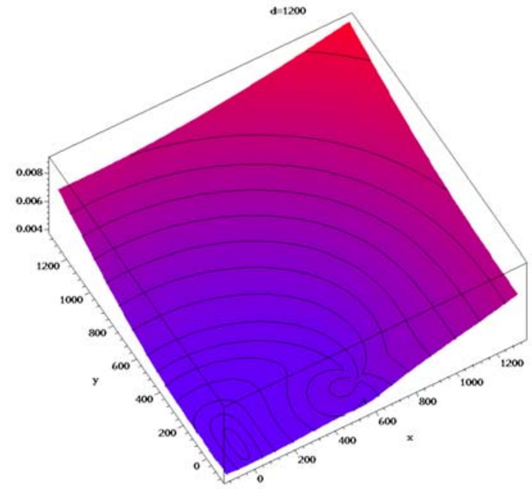

Figure 7

The example we analysed shows that for a typical geodetic totalstation, up to 11 different "optimal" locations can be distinguished as a function of the distance to measure.

The current technological evolution leads to ever smaller values of *b*, but the value of *c* tends to stagnate. If this trend continues, the ratio *c/b* will increase further and the exotic case IV will more often be optimal for medium-range distances.

6. OPTIMAL ANGLE MEASUREMENT

A related problem is the determination of the optimal point *M* for measuring the azimuth of *AB* (i.e. the angle α between *AB* and a reference direction). One measures the distances $d_1=MA$ and $d_2=MB$ along with the angles α_1 and α_2 enclosed by *MA* resp. *MB* and the reference direction. Then α satisfies the equation

$$\tan\alpha = \frac{d_2 \sin\alpha_2 - d_1 \sin\alpha_1}{d_2 \cos\alpha_2 - d_1 \cos\alpha_1} \quad (14)$$

and a similar reasoning to that in §1 leads to the expression

$$\sigma_\alpha^2 = \left(\frac{2T(1+U+T)^2}{(U+T)^2 + 2(T-U)+1} \right) \frac{a^2}{d^2} + \frac{1}{2} Tb^2 + \frac{U+1}{2} c^2$$

When $U=T=0$ (i.e. the middle point of *AB*) the value of this expression is $c^2/2$, and one sees easily that this is the unique minimum value (each term being nonnegative). Consequently, *the optimal location for angle measurement is the middle point, and the corresponding error is* $\sigma_\alpha = c/\sqrt{2}$. (This value of σ_α is smaller than *c* because two independent angle measurements are combined in (14).)

7. CONCLUSION

We have established a closed-form expression for the variance of an eccentric electronic distance measurement using a total station in surveying, and we have compared the values of this variance at the different stationary points of this expression as well as the variance value for a direct electronic measurement, to obtain a full classification of the cases in which each different kind of measurement (direct, eccentric from the middle point, eccentric on the middle perpendicular, or eccentric away from the middle perpendicular) is optimal. This classification only requires polynomial inequalities and, in one case, the solution of a cubic equation. Therefore, this optimisation problem is solved completely. Finally, we have also used the same methods to determine the position for optimal angle measurement, which turns out always to be the middle point.

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BIOGRAPHICAL NOTES

Alain de Wulf has been Professor in Surveying and Geomatic Engineering in the Department of Geography at Ghent University since 1995. He obtained a Ph.D. in Engineering in 1993, and previously obtained a M.Sc. in Civil Engineering and one in Industrial Management. He also graduated with a degree in Computer Science from the same university. He is (co-)author of many research papers in both hydrography and accuracy of classical surveying.

Denis Constales has been a Postdoctoral Researcher in the Department of Mathematical Analysis at Ghent University since 1995. He obtained a Ph.D. in Mathematics in 1989 and previously graduated with a degree in Mathematics from the same University. He is (co-)author of more than thirty research papers in pure and applied mathematics, and was head of the Organizing Committee of the 2002 International Conference on Mathematics in Chemical Kinetics and Engineering (Mackie-2002).

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